

T.3 Minimalpolynome

Die Tabellen T.4 und T.5 enthalten Minimalpolynome, ebenfalls ausgedrückt durch die elementaren symmetrischen Funktionen. Die Bezeichnung rührt daher, daß ihre Werte für alle Variablen a, b, c, \dots stets nichtnegativ sind, aber dennoch „klein“ sein sollen. Aus diesen Identitäten folgen viele Ungleichungen (vgl. Aufgabe U.73).

Tabelle T.4. Minimalpolynome für $n = 3$ ausgedrückt durch die elementaren symmetrischen Funktionen $\sigma_1 \equiv a + b + c$, $\sigma_2 \equiv bc + ca + ab$ und $\sigma_3 \equiv abc$

Polynome 2. Grades

$$\frac{1}{2}Q \equiv \frac{1}{2}[(b-c)^2 + (c-a)^2 + (a-b)^2] = \sigma_1^2 - 3\sigma_2, \quad (\text{T.1})$$

Polynome 3. Grades

$$a(b-c)^2 + b(c-a)^2 + c(a-b)^2 = \sigma_1\sigma_2 - 9\sigma_3, \quad (\text{T.2})$$

$$(b+c)(b-c)^2 + (c+a)(c-a)^2 + (a+b)(a-b)^2 = 2\sigma_1^3 - 7\sigma_1\sigma_2 + 9\sigma_3, \quad (\text{T.3})$$

$$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) = \sigma_1^3 - 4\sigma_1\sigma_2 + 9\sigma_3, \quad (\text{T.4})$$

Polynome 4. Grades

$$\begin{aligned} bc(a-b)(a-c) + ca(b-c)(b-a) + ab(c-a)(c-b) \\ = \frac{1}{2}[a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2] = -3\sigma_1\sigma_3 + \sigma_2^2, \end{aligned} \quad (\text{T.5})$$

$$\begin{aligned} (b-c)^4 + (c-a)^4 + (a-b)^4 = \\ 2[(b-c)^2(c-a)^2 + (c-a)^2(a-b)^2 + (a-b)^2(b-c)^2] \\ = 2\sigma_1^4 - 12\sigma_1^2\sigma_2 + 18\sigma_2^2, \end{aligned} \quad (\text{T.6})$$

$$bc(b-c)^2 + ca(c-a)^2 + ab(a-b)^2 = \sigma_1^2\sigma_2 + 3\sigma_1\sigma_3 - 4\sigma_2^2, \quad (\text{T.7})$$

$$\begin{aligned} a(b+c)(b-c)^2 + b(c+a)(c-a)^2 + c(a+b)(a-b)^2 \\ = \sigma_1^2\sigma_2 - 3\sigma_1\sigma_3 - 2\sigma_2^2, \end{aligned} \quad (\text{T.8})$$

$$\begin{aligned} (b^2+c^2)(b-c)^2 + (c^2+a^2)(c-a)^2 + (a^2+b^2)(a-b)^2 \\ = 2\sigma_1^4 - 10\sigma_1^2\sigma_2 + 6\sigma_1\sigma_3 + 10\sigma_2^2, \end{aligned} \quad (\text{T.9})$$

$$\begin{aligned} a^2(a-b)(a-c) + b^2(b-c)(b-a) + c^2(c-a)(c-b) \\ = \sigma_1^4 - 5\sigma_1^2\sigma_2 + 6\sigma_1\sigma_3 + 4\sigma_2^2, \end{aligned} \quad (\text{T.10})$$

$$(a^2-bc)^2 + (b^2-ca)^2 + (c^2-ab)^2 = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 3\sigma_2^2, \quad (\text{T.11})$$

Polynome 5. Grades

$$\begin{aligned} a(b-c)^4 + b(c-a)^4 + c(a-b)^4 = \\ (b+c)(c-a)^2(a-b)^2 + (c+a)(a-b)^2(b-c)^2 + (a+b)(b-c)^2(c-a)^2 \\ = \sigma_1^3\sigma_2 - 9\sigma_1^2\sigma_3 - 3\sigma_1\sigma_2^2 + 27\sigma_2\sigma_3, \end{aligned} \quad (\text{T.12})$$

$$a^3(b-c)^2 + b^3(c-a)^2 + c^3(a-b)^2 = -4\sigma_1^2\sigma_3 + \sigma_1\sigma_2^2 + 3\sigma_2\sigma_3, \quad (\text{T.13})$$

$$\begin{aligned} (b+c)(b-c)^4 + (c+a)(c-a)^4 + (a+b)(a-b)^4 \\ = 2\sigma_1^5 - 13\sigma_1^3\sigma_2 + 9\sigma_1^2\sigma_3 + 21\sigma_1\sigma_2^2 - 27\sigma_2\sigma_3, \end{aligned} \quad (\text{T.14})$$

$$\begin{aligned} a^2(b+c)(b-c)^2 + b^2(c+a)(c-a)^2 + c^2(a+b)(a-b)^2 \\ = -2\sigma_1^2\sigma_3 + \sigma_1\sigma_2^2 - 3\sigma_2\sigma_3, \end{aligned} \quad (\text{T.15})$$

$$\begin{aligned}
a(b^2 + c^2)(b - c)^2 + b(c^2 + a^2)(c - a)^2 + c(a^2 + b^2)(a - b)^2 \\
= \sigma_1^3 \sigma_2 - 5\sigma_1^2 \sigma_3 - 3\sigma_1 \sigma_2^2 + 15\sigma_2 \sigma_3, \quad (\text{T.17})
\end{aligned}$$

$$\begin{aligned}
(b^3 + c^3)(b - c)^2 + (c^3 + a^3)(c - a)^2 + (a^3 + b^3)(a - b)^2 \\
= 2\sigma_1^5 - 12\sigma_1^3 \sigma_2 + 10\sigma_1^2 \sigma_3 + 17\sigma_1 \sigma_2^2 - 21\sigma_2 \sigma_3, \quad (\text{T.18})
\end{aligned}$$

$$\begin{aligned}
(b + c)(b^2 + c^2)(b - c)^2 + (c + a)(c^2 + a^2)(c - a)^2 + (a + b)(a^2 + b^2)(a - b)^2 \\
= 2\sigma_1^5 - 11\sigma_1^3 \sigma_2 + 11\sigma_1^2 \sigma_3 + 13\sigma_1 \sigma_2^2 - 15\sigma_2 \sigma_3, \quad (\text{T.19})
\end{aligned}$$

$$\begin{aligned}
c(b - c)^2(c - a)^2 + a(c - a)^2(a - b)^2 + b(a - b)^2(b - c)^2 \\
= \sigma_1^5 - 7\sigma_1^3 \sigma_2 + 9\sigma_1^2 \sigma_3 + 12\sigma_1 \sigma_2^2 - 27\sigma_2 \sigma_3, \quad (\text{T.20})
\end{aligned}$$

$$\begin{aligned}
a^3(a - b)(a - c) + b^3(b - c)(b - a) + c^3(c - a)(c - b) \\
= \sigma_1^5 - 6\sigma_1^3 \sigma_2 + 7\sigma_1^2 \sigma_3 + 8\sigma_1 \sigma_2^2 - 12\sigma_2 \sigma_3, \quad (\text{T.21})
\end{aligned}$$

$$\begin{aligned}
a(a^2 - b^2)(a^2 - c^2) + b(b^2 - c^2)(b^2 - a^2) + c(c^2 - a^2)(c^2 - b^2) \\
= \sigma_1^5 - 5\sigma_1^3 \sigma_2 + 7\sigma_1^2 \sigma_3 + 4\sigma_1 \sigma_2^2 - 3\sigma_2 \sigma_3, \quad (\text{T.22})
\end{aligned}$$

Polynome 6. Grades

$$\begin{aligned}
(b - c)^6 + (c - a)^6 + (a - b)^6 \\
= 2\sigma_1^6 - 18\sigma_1^4 \sigma_2 - 12\sigma_1^3 \sigma_3 + 57\sigma_1^2 \sigma_2^2 + 54\sigma_1 \sigma_2 \sigma_3 - 66\sigma_2^3 - 81\sigma_3^2, \quad (\text{T.23})
\end{aligned}$$

$$a^2(b - c)^4 + b^2(c - a)^4 + c^2(a - b)^4 = -2\sigma_1^3 \sigma_3 + \sigma_1^2 \sigma_2^2 - 2\sigma_2^3 + 27\sigma_3^2, \quad (\text{T.24})$$

$$\begin{aligned}
bc(b - c)^4 + ca(c - a)^4 + ab(a - b)^4 \\
= \sigma_1^4 \sigma_2 + 7\sigma_1^3 \sigma_3 - 8\sigma_1^2 \sigma_2^2 - 27\sigma_1 \sigma_2 \sigma_3 + 16\sigma_2^3 + 27\sigma_3^2, \quad (\text{T.25})
\end{aligned}$$

$$\begin{aligned}
a(b + c)(b - c)^4 + b(c + a)(c - a)^4 + c(a + b)(a - b)^4 \\
= \sigma_1^4 \sigma_2 - 7\sigma_1^3 \sigma_3 - 4\sigma_1^2 \sigma_2^2 + 27\sigma_1 \sigma_2 \sigma_3 + 2\sigma_2^3 - 27\sigma_3^2, \quad (\text{T.26})
\end{aligned}$$

$$\begin{aligned}
a^4(b - c)^2 + b^4(c - a)^2 + c^4(a - b)^2 = \\
- 4\sigma_1^3 \sigma_3 + \sigma_1^2 \sigma_2^2 + 10\sigma_1 \sigma_2 \sigma_3 - 2\sigma_2^3 - 9\sigma_3^2, \quad (\text{T.27})
\end{aligned}$$

$$\begin{aligned}
a^3(b + c)(b - c)^2 + b^3(c + a)(c - a)^2 + c^3(a + b)(a - b)^2 \\
= -7\sigma_1 \sigma_2 \sigma_3 + 2\sigma_2^3 + 9\sigma_3^2, \quad (\text{T.28})
\end{aligned}$$

$$\begin{aligned}
(b^2 + c^2)(b - c)^4 + (c^2 + a^2)(c - a)^4 + (a^2 + b^2)(a - b)^4 \\
= 2\sigma_1^6 - 16\sigma_1^4 \sigma_2 + 2\sigma_1^3 \sigma_3 + 41\sigma_1^2 \sigma_2^2 - 34\sigma_2^3 - 27\sigma_3^2, \quad (\text{T.29})
\end{aligned}$$

$$\begin{aligned}
(b - c)^2[(c - a)^4 + (a - b)^4] + (c - a)^2[(a - b)^4 + (b - c)^4] \\
+ (a - b)^2[(b - c)^4 + (c - a)^4] \\
= 2\sigma_1^6 - 18\sigma_1^4 \sigma_2 + 12\sigma_1^3 \sigma_3 + 51\sigma_1^2 \sigma_2^2 - 54\sigma_1 \sigma_2 \sigma_3 - 42\sigma_2^3 + 81\sigma_3^2, \quad (\text{T.30})
\end{aligned}$$

$$(b - c)^2(c - a)^2(a - b)^2 = -4\sigma_1^3 \sigma_3 + \sigma_1^2 \sigma_2^2 + 18\sigma_1 \sigma_2 \sigma_3 - 4\sigma_2^3 - 27\sigma_3^2, \quad (\text{T.31})$$

$$\begin{aligned}
c^2(b - c)^2(c - a)^2 + a^2(c - a)^2(a - b)^2 + b^2(a - b)^2(b - c)^2 \\
= \sigma_1^6 - 8\sigma_1^4 \sigma_2 + 10\sigma_1^3 \sigma_3 + 18\sigma_1^2 \sigma_2^2 - 36\sigma_1 \sigma_2 \sigma_3 - 8\sigma_2^3 + 27\sigma_3^2, \quad (\text{T.32})
\end{aligned}$$

$$\begin{aligned}
& + (a^2 + b^2)(b - c)^2(c - a)^2 \\
& = -10\sigma_1^3\sigma_3 + 3\sigma_1^2\sigma_2^2 + 36\sigma_1\sigma_2\sigma_3 - 10\sigma_2^3 - 27\sigma_3^2, \quad (\text{T.33})
\end{aligned}$$

$$\begin{aligned}
& a^4(a - b)(a - c) + b^4(b - c)(b - a) + c^4(c - a)(c - b) \\
& = \sigma_1^6 - 7\sigma_1^4\sigma_2 + 8\sigma_1^3\sigma_3 + 13\sigma_1^2\sigma_2^2 - 22\sigma_1\sigma_2\sigma_3 - 4\sigma_2^3 + 9\sigma_3^2, \quad (\text{T.34})
\end{aligned}$$

$$\begin{aligned}
& a^2(a^2 - b^2)(a^2 - c^2) + b^2(b^2 - c^2)(b^2 - a^2) + c^2(c^2 - a^2)(c^2 - b^2) \\
& = \sigma_1^6 - 6\sigma_1^4\sigma_2 + 8\sigma_1^3\sigma_3 + 8\sigma_1^2\sigma_2^2 - 16\sigma_1\sigma_2\sigma_3 + 9\sigma_3^2, \quad (\text{T.35})
\end{aligned}$$

$$\begin{aligned}
& bc(a^2 - b^2)(a^2 - c^2) + ca(b^2 - c^2)(b^2 - a^2) + ab(c^2 - a^2)(c^2 - b^2) \\
& = \sigma_1^3\sigma_3 - 7\sigma_1\sigma_2\sigma_3 + \sigma_2^3 + 9\sigma_3^2, \quad (\text{T.36})
\end{aligned}$$

$$\begin{aligned}
& a^2(a^2 - bc) + b^2(b^2 - ca) + c^2(c^2 - ab) \\
& = \sigma_1^6 - 6\sigma_1^4\sigma_2 + 4\sigma_1^3\sigma_3 + 9\sigma_1^2\sigma_2^2 - 6\sigma_1\sigma_2\sigma_3 - 2\sigma_2^3. \quad (\text{T.37})
\end{aligned}$$