$\boxed{\textsf{U.18}}$ Man weise folgende Identitäten nach:

$$\mathcal{M}_n^r(a_1, a_2, \ldots, a_n) = [\mathcal{A}_n(a_1^r, a_2^r, \ldots, a_n^r)]^{\frac{1}{r}}, \tag{U.28}$$

$$\mathcal{G}_n(a_1, a_2, \ldots, a_n) = \mathrm{e}^{\mathcal{A}_n(\ln a_1, \ln a_2, \ldots, \ln a_n)}, \tag{U.29}$$

$$\mathcal{M}_n^{-r}(a_1, a_2, \ldots, a_n) = \frac{1}{\mathcal{M}_n^r\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right)}, \tag{U.30}$$

$$\mathcal{M}_n^{rs}(a_1, a_2, \ldots, a_n) = [\mathcal{M}_n^s(a_1^r, a_2^r, \ldots, a_n^r)]^{\frac{1}{r}}, \tag{U.31}$$

$$\lim_{r \to 0} \mathcal{M}_n^r(a_1, a_2, \ldots, a_n) = \mathcal{G}_n(a_1, a_2, \ldots, a_n). \tag{U.32}$$

U.18 Für die ersten vier Identitäten genügen folgende Zeilen:

$$[\mathcal{A}_n(a_1^r, a_2^r, \ldots, a_n^r)]^{\frac{1}{r}} = \left(\frac{a_1^r + a_2^r + \cdots + a_n^r}{n}\right)^{\frac{1}{r}} \equiv \mathcal{M}_n^r(a_1, a_2, \ldots, a_n),$$

$$\mathrm{e}^{\mathcal{A}_n(\ln a_1, \ln a_2, \ldots, \ln a_n)} = \mathrm{e}^{\frac{\ln a_1 + \ln a_2 + \cdots + \ln a_n}{n}} = (a_1 a_2 \cdots a_n)^{\frac{1}{n}} \equiv \mathcal{G}_n(a_1, a_2, \ldots, a_n),$$

$$\frac{1}{\mathcal{M}_n^r\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right)} = \left(\frac{a_1^{-r} + a_2^{-r} + \cdots + a_n^{-r}}{n}\right)^{-\frac{1}{r}} = \mathcal{M}_n^{-r}(a_1, a_2, \ldots, a_n),$$

$$[\mathcal{M}_n^s(a_1^r, a_2^r, \ldots, a_n^r)]^{\frac{1}{r}} = \left(\frac{a_1^{rs} + a_2^{rs} + \cdots + a_n^{rs}}{n}\right)^{\frac{1}{rs}} = \mathcal{M}_n^{rs}(a_1, a_2, \ldots, a_n).$$

Daß $\mathcal{G}_n$ der Grenzwert von $\mathcal{M}_n^r$ für $r \to 0$ ist, beweisen wir mit der Regel von L'Hospital:

$$\begin{aligned}
\ln\left(\lim_{r\to 0} \mathcal{M}_n^r\right) &= \lim_{r\to 0} (\ln \mathcal{M}_n^r) = \lim_{r\to 0} \frac{\ln\left(\frac{a_1^r + a_2^r + \cdots + a_n^r}{n}\right)}{r} = \frac{0}{0} = \lim_{r\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}r} \ln\left(\frac{a_1^r + a_2^r + \cdots + a_n^r}{n}\right)}{1} \\
&= \lim_{r\to 0} \frac{\frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}r}\left(a_1^r + a_2^r + \cdots + a_n^r\right)}{\frac{a_1^r + a_2^r + \cdots + a_n^r}{n}} = \lim_{r\to 0} \frac{a_1^r \ln a_1 + a_2^r \ln a_2 + \cdots + a_n^r \ln a_n}{a_1^r + a_2^r + \cdots + a_n^r} \\
&= \frac{\ln a_1 + \ln a_2 + \cdots + \ln a_n}{n} = \frac{1}{n} \ln(a_1 a_2 \cdots a_n) = \ln \mathcal{G}_n,
\end{aligned}$$

also $\mathcal{G}_n = \mathcal{M}_n^0$. $\square$