T.1.2 Längen- und Flächenbeziehungen

$$\sigma_1 \equiv a+b+c = 2s,$$

$$\sigma_2 \equiv bc+ca+ab = s^2+4Rr+r^2 = 2\Delta\left(\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}+\frac{1}{\sin\gamma}\right) = 2R(h_a+h_b+h_c),$$

$$\sigma_3 \equiv abc = 4Rrs = 4R\Delta,$$

$$s_2 \equiv a^2+b^2+c^2 = 2(s^2-4Rr-r^2) = 2R(a\sin\alpha+b\sin\beta+c\sin\gamma),$$

$$s_3 \equiv a^3+b^3+c^3 = 2s(s^2-6Rr-3r^2),$$

$$(a+b)(b+c)(c+a) = 2s(s^2+2Rr+r^2),$$

$$(bc)^2+(ca)^2+(ab)^2 = 8\Delta^2+\frac{1}{2}(a^4+b^4+c^4) = \frac{4\Delta^2}{\sin^2\omega},$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{s^2+4Rr+r^2}{4Rrs},$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = \left(\frac{s^2-4Rr+r^2}{4Rrs}\right)^2-\frac{1}{Rr},$$

$$\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab} = \frac{2s}{abc} = \frac{1}{2Rr},$$

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c} = \frac{s^2-2Rr+r^2}{2Rr},$$

$$\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c} = \frac{4R-2r}{r},$$

$$\frac{a^2}{s-a}+\frac{b^2}{s-b}+\frac{c^2}{s-c} = \frac{4s\,(R-r)}{r},$$

$$\frac{a}{(s-b)(s-c)}+\frac{b}{(s-c)(s-a)}+\frac{c}{(s-a)(s-b)} = \frac{2\,(4R+r)}{rs},$$

$$\frac{a^2}{(s-b)(s-c)}+\frac{b^2}{(s-c)(s-a)}+\frac{c^2}{(s-a)(s-b)} = \frac{4\,(R+r)}{r},$$

$$OH^2 = 9R^2-(a^2+b^2+c^2) = R^2(1-8\cos\alpha\cos\beta\cos\gamma),$$

$$OI^2 = R^2-2Rr,$$

$$IH^2 = 2r^2-4R^2\cos\alpha\cos\beta\cos\gamma = 4R^2+4Rr+3r^2-s^2,$$

$$GI^2 = \frac{2}{3}OI^2+\frac{1}{3}IH^2+\frac{2}{9}OH^2 = \frac{1}{9}(s^2-16Rr+5r^2),$$

$$GH = 2\,OG,$$

$$\begin{aligned}[OIH] &= 2R^2\sin\frac{\beta-\gamma}{2}\sin\frac{\alpha-\gamma}{2}\sin\frac{\alpha-\beta}{2}\\ &= \frac{(b-c)(a-c)(a-b)}{8r} \qquad (a>b>c).\end{aligned}$$