T.3 Minimalpolynome

Die Tabellen T.4 und T.5 enthalten Minimalpolynome, ebenfalls ausgedrückt durch die elementaren symmetrischen Funktionen. Die Bezeichnung rührt daher, daß ihre Werte für alle Variablen $a, b, c, \ldots$ stets nichtnegativ sind, aber dennoch „klein“ sein sollen. Aus diesen Identitäten folgen viele Ungleichungen (vgl. Aufgabe U.73).

Tabelle T.4. Minimalpolynome für $n = 3$ ausgedrückt durch die elementaren symmetrischen Funktionen $\sigma_1 \equiv a+b+c$, $\sigma_2 \equiv bc+ca+ab$ und $\sigma_3 \equiv abc$

Polynome 2. Grades

$$\frac{1}{2}Q \equiv \frac{1}{2}\left[(b-c)^2+(c-a)^2+(a-b)^2\right] = \sigma_1^2 - 3\sigma_2, \tag{T.1}$$

Polynome 3. Grades

$$a(b-c)^2+b(c-a)^2+c(a-b)^2 = \sigma_1\sigma_2 - 9\sigma_3, \tag{T.2}$$

$$(b+c)(b-c)^2+(c+a)(c-a)^2+(a+b)(a-b)^2 = 2\sigma_1^3 - 7\sigma_1\sigma_2 + 9\sigma_3, \tag{T.3}$$

$$a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b) = \sigma_1^3 - 4\sigma_1\sigma_2 + 9\sigma_3, \tag{T.4}$$

Polynome 4. Grades

$$bc(a-b)(a-c)+ca(b-c)(b-a)+ab(c-a)(c-b)$$
$$= \frac{1}{2}\left[a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2\right] = -3\sigma_1\sigma_3 + \sigma_2^2, \tag{T.5}$$

$$(b-c)^4+(c-a)^4+(a-b)^4 =$$
$$2\left[(b-c)^2(c-a)^2+(c-a)^2(a-b)^2+(a-b)^2(b-c)^2\right]$$
$$= 2\sigma_1^4 - 12\sigma_1^2\sigma_2 + 18\sigma_2^2, \tag{T.6}$$

$$bc(b-c)^2+ca(c-a)^2+ab(a-b)^2 = \sigma_1^2\sigma_2 + 3\sigma_1\sigma_3 - 4\sigma_2^2, \tag{T.7}$$

$$a(b+c)(b-c)^2+b(c+a)(c-a)^2+c(a+b)(a-b)^2$$
$$= \sigma_1^2\sigma_2 - 3\sigma_1\sigma_3 - 2\sigma_2^2, \tag{T.8}$$

$$(b^2+c^2)(b-c)^2+(c^2+a^2)(c-a)^2+(a^2+b^2)(a-b)^2$$
$$= 2\sigma_1^4 - 10\sigma_1^2\sigma_2 + 6\sigma_1\sigma_3 + 10\sigma_2^2, \tag{T.9}$$

$$a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b)$$
$$= \sigma_1^4 - 5\sigma_1^2\sigma_2 + 6\sigma_1\sigma_3 + 4\sigma_2^2, \tag{T.10}$$

$$(a^2-bc)^2+(b^2-ca)^2+(c^2-ab)^2 = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 3\sigma_2^2, \tag{T.11}$$

Polynome 5. Grades

$$a(b-c)^4+b(c-a)^4+c(a-b)^4 =$$
$$(b+c)(c-a)^2(a-b)^2+(c+a)(a-b)^2(b-c)^2+(a+b)(b-c)^2(c-a)^2$$
$$= \sigma_1^3\sigma_2 - 9\sigma_1^2\sigma_3 - 3\sigma_1\sigma_2^2 + 27\sigma_2\sigma_3, \tag{T.12}$$

$$a^3(b-c)^2+b^3(c-a)^2+c^3(a-b)^2 = -4\sigma_1^2\sigma_3 + \sigma_1\sigma_2^2 + 3\sigma_2\sigma_3, \tag{T.13}$$

$$(b+c)(b-c)^4+(c+a)(c-a)^4+(a+b)(a-b)^4$$
$$= 2\sigma_1^5 - 13\sigma_1^3\sigma_2 + 9\sigma_1^2\sigma_3 + 21\sigma_1\sigma_2^2 - 27\sigma_2\sigma_3, \tag{T.14}$$

$$a^2(b+c)(b-c)^2+b^2(c+a)(c-a)^2+c^2(a+b)(a-b)^2$$
$$= -2\sigma_1^2\sigma_3 + \sigma_1\sigma_2^2 - 3\sigma_2\sigma_3, \tag{T.15}$$

$$bc(b+c)(b-c)^2+ca(c+a)(c-a)^2+ab(a+b)(a-b)^2$$

$$= \sigma_1^3\sigma_2 + \sigma_1^2\sigma_3 - 4\sigma_1\sigma_2^2 + 6\sigma_2\sigma_3, \tag{T.16}$$

$$a(b^2+c^2)(b-c)^2 + b(c^2+a^2)(c-a)^2 + c(a^2+b^2)(a-b)^2$$
$$= \sigma_1^3\sigma_2 - 5\sigma_1^2\sigma_3 - 3\sigma_1\sigma_2^2 + 15\sigma_2\sigma_3, \tag{T.17}$$

$$(b^3+c^3)(b-c)^2 + (c^3+a^3)(c-a)^2 + (a^3+b^3)(a-b)^2$$
$$= 2\sigma_1^5 - 12\sigma_1^3\sigma_2 + 10\sigma_1^2\sigma_3 + 17\sigma_1\sigma_2^2 - 21\sigma_2\sigma_3, \tag{T.18}$$

$$(b+c)(b^2+c^2)(b-c)^2 + (c+a)(c^2+a^2)(c-a)^2 + (a+b)(a^2+b^2)(a-b)^2$$
$$= 2\sigma_1^5 - 11\sigma_1^3\sigma_2 + 11\sigma_1^2\sigma_3 + 13\sigma_1\sigma_2^2 - 15\sigma_2\sigma_3, \tag{T.19}$$

$$c(b-c)^2(c-a)^2 + a(c-a)^2(a-b)^2 + b(a-b)^2(b-c)^2$$
$$= \sigma_1^5 - 7\sigma_1^3\sigma_2 + 9\sigma_1^2\sigma_3 + 12\sigma_1\sigma_2^2 - 27\sigma_2\sigma_3, \tag{T.20}$$

$$a^3(a-b)(a-c) + b^3(b-c)(b-a) + c^3(c-a)(c-b)$$
$$= \sigma_1^5 - 6\sigma_1^3\sigma_2 + 7\sigma_1^2\sigma_3 + 8\sigma_1\sigma_2^2 - 12\sigma_2\sigma_3, \tag{T.21}$$

$$a(a^2-b^2)(a^2-c^2) + b(b^2-c^2)(b^2-a^2) + c(c^2-a^2)(c^2-b^2)$$
$$= \sigma_1^5 - 5\sigma_1^3\sigma_2 + 7\sigma_1^2\sigma_3 + 4\sigma_1\sigma_2^2 - 3\sigma_2\sigma_3, \tag{T.22}$$

Polynome 6. Grades

$$(b-c)^6 + (c-a)^6 + (a-b)^6$$
$$= 2\sigma_1^6 - 18\sigma_1^4\sigma_2 - 12\sigma_1^3\sigma_3 + 57\sigma_1^2\sigma_2^2 + 54\sigma_1\sigma_2\sigma_3 - 66\sigma_2^3 - 81\sigma_3^2, \tag{T.23}$$

$$a^2(b-c)^4 + b^2(c-a)^4 + c^2(a-b)^4 = -2\sigma_1^3\sigma_3 + \sigma_1^2\sigma_2^2 - 2\sigma_2^3 + 27\sigma_3^2, \tag{T.24}$$

$$bc(b-c)^4 + ca(c-a)^4 + ab(a-b)^4$$
$$= \sigma_1^4\sigma_2 + 7\sigma_1^3\sigma_3 - 8\sigma_1^2\sigma_2^2 - 27\sigma_1\sigma_2\sigma_3 + 16\sigma_2^3 + 27\sigma_3^2, \tag{T.25}$$

$$a(b+c)(b-c)^4 + b(c+a)(c-a)^4 + c(a+b)(a-b)^4$$
$$= \sigma_1^4\sigma_2 - 7\sigma_1^3\sigma_3 - 4\sigma_1^2\sigma_2^2 + 27\sigma_1\sigma_2\sigma_3 + 2\sigma_2^3 - 27\sigma_3^2, \tag{T.26}$$

$$a^4(b-c)^2 + b^4(c-a)^2 + c^4(a-b)^2 =$$
$$- 4\sigma_1^3\sigma_3 + \sigma_1^2\sigma_2^2 + 10\sigma_1\sigma_2\sigma_3 - 2\sigma_2^3 - 9\sigma_3^2, \tag{T.27}$$

$$a^3(b+c)(b-c)^2 + b^3(c+a)(c-a)^2 + c^3(a+b)(a-b)^2$$
$$= -7\sigma_1\sigma_2\sigma_3 + 2\sigma_2^3 + 9\sigma_3^2, \tag{T.28}$$

$$(b^2+c^2)(b-c)^4 + (c^2+a^2)(c-a)^4 + (a^2+b^2)(a-b)^4$$
$$= 2\sigma_1^6 - 16\sigma_1^4\sigma_2 + 2\sigma_1^3\sigma_3 + 41\sigma_1^2\sigma_2^2 - 34\sigma_2^3 - 27\sigma_3^2, \tag{T.29}$$

$$(b-c)^2[(c-a)^4 + (a-b)^4] + (c-a)^2[(a-b)^4 + (b-c)^4]$$
$$+ (a-b)^2[(b-c)^4 + (c-a)^4]$$
$$= 2\sigma_1^6 - 18\sigma_1^4\sigma_2 + 12\sigma_1^3\sigma_3 + 51\sigma_1^2\sigma_2^2 - 54\sigma_1\sigma_2\sigma_3 - 42\sigma_2^3 + 81\sigma_3^2, \tag{T.30}$$

$$(b-c)^2(c-a)^2(a-b)^2 = -4\sigma_1^3\sigma_3 + \sigma_1^2\sigma_2^2 + 18\sigma_1\sigma_2\sigma_3 - 4\sigma_2^3 - 27\sigma_3^2, \tag{T.31}$$

$$c^2(b-c)^2(c-a)^2 + a^2(c-a)^2(a-b)^2 + b^2(a-b)^2(b-c)^2$$

$$= \sigma_1^6 - 8\sigma_1^4\sigma_2 + 10\sigma_1^3\sigma_3 + 18\sigma_1^2\sigma_2^2 - 36\sigma_1\sigma_2\sigma_3 - 8\sigma_2^3 + 27\sigma_3^2, \qquad \text{(T.32)}$$

$$(b^2+c^2)(c-a)^2(a-b)^2 + (c^2+a^2)(a-b)^2(b-c)^2 + (a^2+b^2)(b-c)^2(c-a)^2 = -10\sigma_1^3\sigma_3 + 3\sigma_1^2\sigma_2^2 + 36\sigma_1\sigma_2\sigma_3 - 10\sigma_2^3 - 27\sigma_3^2, \qquad \text{(T.33)}$$

$$a^4(a-b)(a-c) + b^4(b-c)(b-a) + c^4(c-a)(c-b) = \sigma_1^6 - 7\sigma_1^4\sigma_2 + 8\sigma_1^3\sigma_3 + 13\sigma_1^2\sigma_2^2 - 22\sigma_1\sigma_2\sigma_3 - 4\sigma_2^3 + 9\sigma_3^2, \qquad \text{(T.34)}$$

$$a^2(a^2-b^2)(a^2-c^2) + b^2(b^2-c^2)(b^2-a^2) + c^2(c^2-a^2)(c^2-b^2) = \sigma_1^6 - 6\sigma_1^4\sigma_2 + 8\sigma_1^3\sigma_3 + 8\sigma_1^2\sigma_2^2 - 16\sigma_1\sigma_2\sigma_3 + 9\sigma_3^2, \qquad \text{(T.35)}$$

$$bc(a^2-b^2)(a^2-c^2) + ca(b^2-c^2)(b^2-a^2) + ab(c^2-a^2)(c^2-b^2) = \sigma_1^3\sigma_3 - 7\sigma_1\sigma_2\sigma_3 + \sigma_2^3 + 9\sigma_3^2, \qquad \text{(T.36)}$$

$$a^2(a^2-bc) + b^2(b^2-ca) + c^2(c^2-ab) = \sigma_1^6 - 6\sigma_1^4\sigma_2 + 4\sigma_1^3\sigma_3 + 9\sigma_1^2\sigma_2^2 - 6\sigma_1\sigma_2\sigma_3 - 2\sigma_2^3. \qquad \text{(T.37)}$$